MECE 3320 – Measurements & Instrumentation

Uncertainty Analysis

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Introduction

Errors are a property of the measurement

- Repeatability
- Hysteresis
- Linearity
- Sensitivity
- Zero shift etc..

Uncertainty analysis is the process of identifying, quantifying and combining the errors.

Measurement errors can be grouped into two categories – *Random & Systematic errors*



Figure 5.1. Distribution of errors upon repeated measurements.

Stages in Uncertainty Analysis

There are different stages in an uncertainty analysis:

- Design stage
- Single measurement
- Multiple measurement

Design Stage Uncertainty Analysis: Initial analysis performed prior to measurement. This uncertainty is based on the resolution of the instrument to be used assuming that all other sources of error are zero. This is called *zero-order uncertainty* (u_0) .



Design Stage Uncertainty Analysis

EXAMPLE 5.2

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A voltmeter is to be used to measure the output from a pressure transducer as an electrical signal. The nominal pressure is expected to be about 3 psi (3 lb/in.²). Estimate the design-stage uncertainty in this combination. The following information is available:

oltmeter	
Resolution:	10 µV
Accuracy:	within 0.001% of reading
fransducer	
Range:	±5 psi
Sensitivity:	1 V/psi
Input power:	10 VDC $\pm 1\%$
Output:	±5 V
Linearity:	within 2.5 mV/psi over range
Sensitivity:	within 2 mV/psi over range
Resolution:	negligible

KNOWN Instrument specifications

ASSUMPTIONS Values representative of instrument at 95% probability

FIND u_c for each device and u_d for the measurement system

Design Stage Uncertainty Analysis

SOLUTION The procedure in Figure 5.2 will be used for both instruments to estimate the designstage uncertainty in each. The resulting uncertainties will then be combined using the RSS approximation to estimate the system u_d .

The uncertainty in the voltmeter at the design stage is given by equation (5.3) as

$$(u_d)_E = \pm \sqrt{(u_o)_E^2 + (u_c)_E^2}$$

From the information available,

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$$(u_0)_E = \pm 5 \,\mu \text{V} \quad (95\%)$$

For a nominal pressure of 3 psi, we expect to measure an output of 3 V. Then,

$$(u_c)_E = \pm (3 \text{ V} \times 0.00001) = \pm 30 \,\mu\text{V}$$
 (95% assumed)

so that the design-stage uncertainty in the voltmeter is

$$(u_d)_E = \pm 30.4 \,\mu \text{V} \quad (95\%)$$

Design Stage Uncertainty Analysis

The uncertainty in the pressure transducer output at the design stage is also given by equation (5.17). Assuming that we operate within the input power range specified, the instrument output uncertainty can be estimated by considering the uncertainty in each of the instrument elemental errors of linearity, e_1 , and sensitivity, e_2 :

$$(u_c)_p = \sqrt{e_1^2 + e_2^2}$$
 (95% assumed)
= $\pm \sqrt{(2.5 \text{ mV/psi} \times 3 \text{ psi})^2 + (2 \text{ mV/psi} \times 3 \text{ psi})^2}$
= $\pm 9.61 \text{ mV}$ (95%)

Since $(u_0) \approx 0$ V/psi, then the design-stage uncertainty in the transducer in terms of indicated voltage is $(u_d)_p = \pm 9.61$ mV (95%).

Finally, u_d for the combined system is found by use of the RSS method using the design-stage uncertainties of the two devices. The design-stage uncertainty in pressure as indicated by this measurement system is estimated to be

$$u_d = \pm \sqrt{(u_d)_E^2 + (u_d)_p^2}$$

= $\pm \sqrt{(0.030 \text{ mV})^2 + (9.61 \text{ mV})^2}$
= $\pm 9.61 \text{ mV}$ (95%)

But since the sensitivity is 1 V/psi, the uncertainty in pressure is better stated as $u_d = \pm 0.0096 \, \text{psi}$ (95%)

Error Sources

Design Stage Uncertainty Analysis provides information and assess methodology for instrument selection but cannot provide the sources of error that influence a measurement.

So, what are the *sources of error* that we need to know to carry out an uncertainty analysis?

• *Calibration Errors*: Errors that enter the measurement system during calibration

• *Data Acquisition Errors*: Errors that include sensor and instrument errors, uncontrolled variables such as changes in operating conditions, installation effects and measured variable spatio-temporal variations.

• Data Reduction Errors: Basically these are the errors due to curve-fits and correlations

Once we know the sources of error, how do we quantify these errors?

Measurement error = Systematic (Bias) + Random (precision) Error

Systematic and Random Error

Systematic error

If you ask the person who sells fish at your favorite market to weight a piece of fish several times for you, and he puts his thumb on the scale in a way that makes the fish seem 2 ounces heavier than it is, that's bias- a systematic tendency to over or underestimate the true value. Notice that systematic error doesn't move around from observation to observation- that's what makes it systematic.

Random Error

They have no patterns or trends and their average is close to zero. Weigh your fish several times using different weights, you will get different answers because of the random errors.

Error Propagation

Estimate

filling time

Measure tank

Calculate hose

discharge rate

Measure bucket volume

volume

Task: Estimate the filling time of a water tank using a hose of a given diameter

Need to know the discharge rate from the hose to estimate the filling time



Measure time

Error Propagation

Measure the time required to fill up a bucket of known volume.

Error Propagation

EXAMPLE 5.3

For a displacement transducer having a calibration curve y = KE, estimate the uncertainty in displacement y for E = 5.00 V, if K = 10.10 mm/V with $u_K = \pm 0.10$ mm/V and $u_E = \pm 0.01$ V at 95% confidence.

KNOWN

$$y = KE$$

 $E = 5.00 V \quad u_E = \pm 0.01 V$
 $K = 10.10 \text{ mm/V} \quad u_K = \pm 0.10 \text{ mm/V}$

FIND

u_y

SOLUTION Based on equations (5.10) and (5.11), respectively,

$$\overline{y} = f(\overline{E}, \overline{K})$$
 and $u_y = f(u_E, u_K)$

From equation (5.13), the uncertainty in the displacement at y = KE is

$$u_y = \pm \left[\left(\theta_E u_E \right)^2 + \left(\theta_K u_K \right)^2 \right]^{1/2}$$

where the sensitivity indices are evaluated from equation (5.12) as

$$\theta_E = \frac{\partial y}{\partial E} = K \quad \text{and} \quad \theta_K = \frac{\partial y}{\partial K} = E$$

Error Propagation

or we can write equation (5.13) as

$$u_y = \pm \left[\left(K u_E \right)^2 + \left(E u_K \right)^2 \right]^{1/2}$$

The operating point occurs at the nominal or the mean values of E = 5.00 V and y = 50.50 mm. With E = 5.00 V and K = 10.10 mm/V and substituting for u_E and u_K , evaluate u_y at its operating point:

$$u_{y|_{y=50.5}} = \pm \left[(0.10)^2 + (0.5)^2 \right]^{1/2} = \pm 0.51 \,\mathrm{mm} \quad (95\%)$$

Single/Multiple Measurement Uncertainty Analysis



Figure 5.6. Multiple-measurement uncertainty procedure for combining uncertainties.

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Single/Multiple Measurement Uncertainty Analysis

EXAMPLE 5.8

Ten repeated measurements of force, F, are made over time under fixed operating conditions. The data are listed below. Estimate the standard random uncertainty due to the elemental error in the mean value of the force that is introduced into the measured data through the data scatter.

F[N]	n	<i>F</i> [N]
123.2	6	119.8
115.6	7	117.5
117.1	8	120.6
125.7	9	118.8
121.1	10	121.9
	F[N] 123.2 115.6 117.1 125.7 121.1	F[N] n 123.2 6 115.6 7 117.1 8 125.7 9 121.1 10

KNOWN Measured data set

N = 10

ASSUMPTIONS Error due to data scatter only

FIND Estimate P_1 [equation (5.20)]

SOLUTION The mean value of the force based on this finite data set is computed from equation (4.14) as $\overline{F} = 120.1$ N. A random error is associated with the estimate of the mean value because of data scatter. This error enters the measurement during data acquisition (Table 5.2). The standard random uncertainty in this error can be computed through the standard deviation of the means, equation (4.16):

$$P_1 = S_{\overline{F}} = \frac{S_F}{\sqrt{N}}$$
$$= \frac{3.2}{\sqrt{10}} = 1.01$$

N

Single/Multiple Measurement Uncertainty Analysis

EXAMPLE 5.12

After an experiment to measure stress in a loaded beam, an uncertainty analysis reveals the following values of uncertainty in stress measurement whose magnitudes were computed from elemental errors using equations (5.20) and (5.21).

$B_1 = 1.0 \text{N/cm}^2$	$B_2 = 2.1 \mathrm{N/cm^2}$	$B_3 = 0 N/cm^2$
$P_1 = 4.6 \text{N}/\text{cm}^2$	$P_2 = 10.3 \mathrm{N/cm^2}$	$P_3 = 1.2 \mathrm{N/cm^2}$
$v_1 = 14$	$v_2 = 37$	$v_3 = 8$

If the mean value of the stress in the measurement is $\overline{\sigma} = 223.4 \text{ N/cm}^2$, determine the best estimate of the stress at a 95% confidence level, assuming all errors are accounted for.

KNOWN Experimental errors

ASSUMPTIONS All elemental errors (K = 3) have been included.

- FIND P, B, and u_{σ} (using equations 5.20–5.23)
- **SOLUTION** We seek values for the statement, $\sigma' = \overline{\sigma} \pm u_{\sigma}(95\%)$, given that $\overline{\sigma} = 223.4 \text{ N/cm}^2$. The uncertainty estimate in the measurement is obtained through equations (5.20) through (5.23). The measurement standard random uncertainty is given by equation (5.20) as

$$P = \left(P_1^2 + P_2^2 + P_2^2\right)^{1/2} = 11.3 \,\mathrm{N/cm^2}$$

Single/Multiple Measurement Uncertainty Analysis

The measurement systematic uncertainty is given by equation (5.21) as

$$B = (B_1^2 + B_2^2 + B_2^2)^{1/2} = 2.3 \,\mathrm{N/cm^2}$$

The degrees of freedom in P is found from equation (5.23) to be

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$$\mathbf{v} = \frac{\left(\sum_{k=1}^{3} P_k^2\right)^2}{\sum_{k=1}^{3} P_k^4 / \mathbf{v}_k} \approx 49$$

Therefore, the *t* estimator is $t_{49,95} \sim 2.0$. The uncertainty estimate is found using equation (5.22) to be

$$u_{\sigma} = \pm \left[B^2 + (t_{v,95}P)^2 \right]^{1/2} = \pm \left[\left(2.3 \,\text{N/cm}^2 \right)^2 + \left(2 \times 11.3 \,\text{N/cm}^2 \right)^2 \right]^{1/2} \\ = \pm 22.7 \,\text{N/cm}^2$$

The best estimate is given in the form of equation (4.1) as

$$\sigma' = 223.4 \pm 22.7 \,\mathrm{N/cm^2}$$
 (95%)

Propagation of Uncertainty to a Result

EXAMPLE 5.13

The density of a gas, ρ , which is believed to follow the ideal gas equation of state, $\rho = p/RT$, is to be estimated through separate measurements of pressure, p, and temperature, T. The gas is housed within a rigid impermeable vessel. The literature accompanying the pressure measurement system states an instrument uncertainty to within 1% of the reading, and that accompanying the temperature measuring system suggests 0.6°R. Twenty measurements of pressure, $N_p = 20$, and ten measurements of temperature, $N_T = 10$, are made with the following statistical outcome:

 $\overline{p} = 2253.91 \text{ psfa}$ $S_p = 167.21 \text{ psfa}$ $\overline{T} = 560.4^{\circ}\text{R}$ $S_T = 3.0^{\circ}\text{R}$

where psfa refers to lb/ft^2 absolute. Determine a best estimate of the density. The gas constant is R = 54.7 ft-lb/lb_m-°R.

KNOWN

$$\overline{p}, S_p, \overline{T}, S_T$$

 $\rho = p/RT; R = 54.7 \text{ ft-lb/lb}_{m} - {}^{\circ}R$

ASSUMPTIONS Gas behaves as an ideal gas

FIND

$$\rho' = \overline{\rho} \pm u_{\rho}$$
 (95%)

$$\overline{\rho} = \frac{\overline{p}}{R\overline{T}} = 0.074 \, \mathrm{lb_m/ft^3}$$

Propagation of Uncertainty to a Result

$$(B_{1})_{p} = 22.5 \text{ psfa} \quad (P_{1})_{p} = 0 \qquad (P_{2})_{p} = S_{\overline{p}} = \frac{S_{p}}{\sqrt{N}} = \frac{167.21 \text{ psfa}}{\sqrt{20}} = 37.4 \text{ psfa} \quad v_{p} = 19$$
$$(B_{2})_{p} = 0$$
$$(B_{2})_{p} = 0$$
$$(B_{2})_{T} = 0 \qquad (P_{2})_{T} = S_{\overline{T}} = \frac{S_{T}}{\sqrt{N}} = \frac{3.0^{\circ}\text{R}}{\sqrt{10}} = 0.9^{\circ}\text{R} \quad v_{p} = 9$$
$$(B_{2})_{T} = 0$$

$$(B)_p = \left[(22.5)^2 + (0)^2 \right]^{1/2} = 22.5 \text{ psfa}$$
$$(P)_p = \left[(0)^2 + (37.4)^2 \right]^{1/2} = 37.4 \text{ psfa}$$

Propagation of Uncertainty to a Result

 $(B)_T = 0.6^\circ R$ $(P)_T = 0.9^\circ R$

with degrees of freedom determined from equation (5.23) to be

$$(\mathbf{v})_p = N_p - 1 = 19$$
$$(\mathbf{v})_T = 9$$

The propagation of systematic and random uncertainties through to the result, the density, will be estimated using the RSS of equations (5.27) and (5.28):

$$P_{\rho} = \left[\left(\frac{\partial \rho}{\partial T} P_T \right)^2 + \left(\frac{\partial \rho}{\partial p} P_p \right)^2 \right]^{1/2}$$

= $\left[(1.2 \times 10^{-4})^2 + (1.2 \times 10^{-3})^2 \right]^{1/2}$ (5.31)
= $0.0012 \, \mathrm{lb_m/ft^3}$

Propagation of Uncertainty to a Result

$$B_{\rho} = \left[\left(\frac{\partial \rho}{\partial T} B_T \right)^2 + \left(\frac{\partial \rho}{\partial p} B_p \right)^2 \right]^{1/2}$$
$$= \left[\left(8 \times 10^{-5} \right)^2 + \left(7 \times 10^{-4} \right)^2 \right]^{1/2}$$
$$= 0.0007 \, \mathrm{lb_m/ft^3}$$

The degrees of freedom in the density is determined from equation (5.30):

$$v = \frac{\left[\left(\frac{\partial\rho}{\partial T}P_T\right)^2 + \left(\frac{\partial\rho}{\partial p}P_p\right)^2\right]^2}{\left(\frac{\partial\rho}{\partial T}P_T\right)^4 / v_T + \left(\frac{\partial\rho}{\partial p}P_p\right)^4 / v_p} = 23$$

From Table 4.4, $t_{23,95} = 2.06$.

The uncertainty in the mean value of density is estimated from equation (5.29):

$$u_{\rho} = \left[B_{\rho}^2 + \left(t_{23,95}P_{\rho}\right)^2\right]^{1/2}$$

= 0.0025 lb_m/ft³ (95%)

The best estimate of the density is given in the form of equation (5.24):

$$\rho' = 0.074 \pm 0.0025 \, lb_m/ft^3 \quad (95\%$$

This measurement of density has an uncertainty of about 3.4%.